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Note

On Acceleration of MacCormack's Scheme*

An acceleration of MacCormack's scheme due to Désidéri and Tannehill is analyzed. It is found that for hyperbolic problems one cannot improve upon the efficiency of MacCormack's method. For parabolic problems the time step can be chosen arbitrarily large without loss of stability by an appropriate choice of the acceleration parameters. When applied to the heat equation this method is equivalent to both the Dufort–Frankel scheme and to MSOR.

Désidéri and Tannehill [2] recently introduced an overrelaxation in order to accelerate the convergence of MacCormack's scheme [6] to steady state. They present some impressive computational results. In this note we present some further analysis to explain these results.

To facilitate the analysis we introduce a minor modification of their scheme. Consider the equation

$$u_t = f_x + (Du_x)_x . \tag{1}$$

Then we analyze the difference scheme

$$\begin{aligned} \bar{u}_j^{n+1} &= (1-\bar{\omega})\,\bar{u}_j^{\,n} + \bar{\omega}\left[u_j^{\,n} + \Omega\frac{\Delta t}{\Delta x}(f_{j+1}^n - f_j^n)\right. \\ &\quad \left. + \Omega\frac{\Delta t}{(\Delta x)^2}(D_{j+\frac{1}{2}}^n(u_{j+1}^n - u_j^{\,n}) - D_{j-\frac{1}{2}}^n(u_j^{\,n} - u_{j-1}^n))\right] \end{aligned} \tag{2}$$

and

$$\begin{aligned} u_j^{n+1} &= (1-\omega)\,u_j^{\,n} + \omega\left[\bar{u}_j^{n+1} + \frac{\Omega\Delta t}{\Delta x}(\bar{f}_j^{n+1} - \bar{f}_{j-1}^{n+1})\right. \\ &\quad \left. + \frac{\Omega\Delta t}{(\Delta x)^2}(\bar{D}_{j+\frac{1}{2}}^{n+1}(\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1}) - \bar{D}_{j-\frac{1}{2}}^{n+1}(\bar{u}_j^{n+1} - \bar{u}_{n-1}^{n+1}))\right], \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= (1-\omega)(1-\bar{\omega}), \\ \gamma_2 &= 2\omega\bar{\omega}, \\ \Omega &= (1-\gamma_1)/\gamma_2 . \end{aligned} \tag{3}$$

ω, $\bar{\omega}$ are the acceleration parameters.

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This difference scheme is consistent with the differential equation for all nonzero values of the acceleration parameters. The modified equation obtained by Désidéri and Tannehill [3] is valid only for linear equations. The only way to converge faster to the steady-state solution is by using larger time steps. As Désidéri and Tannehill remark it is difficult to analyze this combined system. Hence, we shall analyze separately the linear hyperbolic and parabolic systems.

For the hyperbolic equation with constant coefficients, i.e., $f = au$, $D = 0$, it follows by a domain-of-dependence argument that one cannot increase the stability limit over MacCormack's scheme by any choice of ω, $\bar{\omega}$. Alternatively, algorithm (2) can be written after a Fourier transform, for this case as

$$\begin{pmatrix} 1 & 0 \\ -\omega[1 + \lambda(1 - e^{-i\alpha})] & 1 \end{pmatrix}\begin{pmatrix} \bar{v}^{n+1} \\ v^{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \bar{\omega} & \bar{\omega}[1 + \lambda(e^{i\alpha} - 1)] \\ 0 & 1 - \omega \end{pmatrix}\begin{pmatrix} \bar{v}^{n} \\ v^{n} \end{pmatrix}, \tag{4}$$

where v is the Fourier transform of u, α is the Fourier variable, and $\lambda = a\Omega\,\Delta t/\Delta x$ or

$$\begin{aligned} Aw^{n+1} &= Bw^n, \\ w^{n+1} &= Gw^n = A^{-1}Bw^n, \end{aligned} \qquad w^n = \begin{pmatrix} \bar{v}^n \\ v^n \end{pmatrix}.$$

Letting g be an eigenvalue of G we have

$$0 = g^2 - g[1 + \gamma_1 + \gamma_2(h(\alpha) - 1)] + \gamma_1 \tag{5}$$

where γ_1, γ_2 are given by (2) and $h(\alpha)$ is the amplification factor for the MacCormack scheme, i.e., $h(\alpha) = 1 + \lambda i \sin\alpha - \lambda^2(1 - \cos\alpha)$.

Using the theorem of Miller [8] we found that necessary and sufficient conditions for g to be in the unit circle are

$$\begin{aligned} &(1) \qquad |\gamma_1| \leqslant 1, \\ &(2) \quad |[1 + \gamma_1 + \gamma_2(\operatorname{Re} h(\alpha) - 1)][1 - \gamma_1] + i\gamma_2 \operatorname{Im} h(\alpha)(1 + \gamma_1)| \leqslant 1 - \gamma_1^2. \end{aligned} \tag{6}$$

This implies that

$$\lambda^2 \leqslant \min_{|\xi| \leqslant 1} \frac{((1 - \gamma_1)^2/(1 + \gamma_1)) - \gamma_2(1 - \xi^2)}{((1 - \gamma_1)^2/(1 + \gamma_1)^2)\,\gamma_2\xi^2}; \qquad \xi = \sin(\alpha/2). \tag{7}$$

Thus, we require that

$$\gamma_2 \leqslant (1 - \gamma_1)^2/(1 + \gamma_1) \tag{8}$$

and

$$a\Delta t/\Delta x \leqslant ((1 + \gamma_1)\gamma_2/(1 - \gamma_1)^2)^{1/2}. \tag{9}$$

Combining Eqs. (8) and (9) we see that $a\Delta t/\Delta x \leqslant 1$; i.e., we cannot improve on the original MacCormack schemes for hyperbolic equations.

For other basic algorithms the overrelaxation can improve the stability. Thus, one can show that this technique will increase the allowable time for some parameters

in the fourth-order extension of MacCormack's method [5]. We observe that Désidéri and Tannehill found their smallest increase in convergence for the inviscid fluid dynamic equations. Even this increase can be explained by two factors. First, Désidéri and Tannehill used the two-dimensional version of MacCormack's scheme. It is known (see, for example, [6] or [9]) that this method is not always stable. Thus, the acceleration may have stabilized the algorithm.

As seen from Eq. (7), whenever Eq. (8) is true the stability requirements come from the long wavelengths rather than the high frequencies. Numerical tests on nozzle flow done by the authors show that one can exceed the Courant condition by 5 to 10% when the flow approaches a steady state. We conjecture that the stabilizing effects of the approach to steady state prevents the amplification of long wavelengths while the high-frequency noise is damped by the scheme. This property may be more important for different values of ω, $\bar{\omega}$ than those for the original MacCormack method.

For the parabolic case, $f = 0$ and D constant, the eigenvalues of the amplification factor are given by

$$0 = g^2 - g[1 + \gamma_1 - \gamma_2(1 - (1 - 4\sigma\xi^2)^2)] + \gamma_1, \tag{10}$$

with γ_1, γ_2, ξ as before and $\sigma = D\Omega\Delta t/(\Delta x)^2$. Again, using the theory of Miller we require

(1) $$|\gamma_1| \leqslant 1,$$

(2) $$|1 + \gamma_1 - \gamma_2[1 - (1 - 4\sigma\xi^2)^2]|\,(1 - \gamma_1) \leqslant (1 - \gamma_1)^2.$$

This implies that

$$D\Delta t/(\Delta x)^2 \leqslant \gamma_2/2(1 - \gamma_1) \tag{11}$$

and if $\gamma_2 > 4(1 + \gamma_1)$ we also require

$$\frac{D\Delta t}{(\Delta x)^2} \leqslant \frac{\gamma_2}{4(1 - \gamma_1)}\left[1 - \left(1 - \frac{4(1 + \gamma_1)}{\gamma_2}\right)^{1/2}\right].$$

We thus see that by choosing γ_1 close to one we can allow arbitrarily large time steps. γ_1, γ_2 are not independent. In order to solve for ω, $\bar{\omega}$ we must have

$$\gamma_2 < 2(1 - (\gamma_1)^{1/2})^2$$

or (12)

$$\gamma_2 > 2(1 + (\gamma_1)^{1/2})^2.$$

To select the optimal acceleration parameters we rewrite Eq. (2) with $f = 0$ and let $\sigma = \Omega(\Delta t/(\Delta x)^2)$:

$$\begin{aligned} v_j^{n+1} &= (1 - \bar{\omega})\, v_j^n + \bar{\omega}[\sigma(D_{j+\frac{1}{2}} u_{j+1}^n + D_{j-\frac{1}{2}} u_{j-1}^n) - (\sigma D_{j+\frac{1}{2}} + \sigma D_{j-\frac{1}{2}} - 1)\, u_j^n], \\ u_j^{n+1} &= (1 - \omega)\, u_j^n + \omega[\sigma(D_{j+\frac{1}{2}} v_{j+1}^{n+1} + D_{j-\frac{1}{2}} v_{j-1}^{n+1}) - (\sigma D_{j+\frac{1}{2}} + \sigma D_{j-\frac{1}{2}} - 1)\, v_j^{n+1}]. \end{aligned} \tag{13}$$

This is identical to MSOR for which there exists a known theory (see [10]). In particular the optimal parameters for the two-dimensional heat equation converging

to the Poisson equation in a square are given by $\omega = \bar{\omega} = 2 - 2h$. For more general domains and nonconstant D the optimal parameters are usually not known a priori.

Désidéri and Tannehill found that the accelerated scheme was more efficient at fine meshes. The reason for this is that the cell Reynolds number decreases with finer meshes, hence making the problem more parabolic-like. Since the scheme mainly accelerates the parabolic terms this effect is well understood.

We now consider the Dufort–Frankel scheme with a stabilizer as given by Gottlieb and Gustafsson [4]:

$$w_j^{n+1} = w_j^{n-1} + \alpha[D_{j+\frac{1}{2}}(w_{j+1}^n - w_j^n) - D_{j-\frac{1}{2}}(w_j^n - w_{j-1}^n)] - \alpha\Gamma(w_j^{n+1} - 2w_j^n + w_j^{n-1}), \quad (14)$$

where $\alpha = 2\Delta t/(\Delta x)^2$ and $\Gamma \geqslant$ spectral radius of D. Since Γ is a scalar even for vector equations we can solve explicitly for w_j^{n+1} with no matrix inversions. We now let $u_j^n = w_j^{2n}$ and $v_j^n = w_j^{2n-1}$. We then find that

$$\begin{aligned} v_j^{n+1} &= \frac{1 - \alpha\Gamma}{1 + \alpha\Gamma} v_j^n + \frac{\alpha}{1 + \alpha\Gamma}[D_{j+\frac{1}{2}}u_{j+1}^n + D_{j-\frac{1}{2}}u_{j-1}^n - (D_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} + \Gamma)u_j^n], \\ u_j^{n+1} &= \frac{1 - \alpha\Gamma}{1 + \alpha\Gamma} u_j^n + \frac{\alpha}{1 + \alpha\Gamma}[D_{j+\frac{1}{2}}v_{j+1}^{n+1} + D_{j-\frac{1}{2}}v_{j-1}^{n+1} - (D_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} + \Gamma)v_j^{n+1}], \end{aligned} \quad (15)$$

but this is identical to Eq. (13) if we identify

$$\omega = 2\alpha\Gamma/(1 + \alpha\Gamma), \qquad \sigma = 1/2\Gamma \qquad \text{or} \qquad \alpha = \omega/(2 - \omega)\Gamma, \qquad \Gamma = 1/2\sigma.$$

Thus, for a pure parabolic problem the accelerated MacCormack scheme is identical to the Dufort–Frankel scheme where the time step of the MacCormack scheme is related to the dissipation of the Dufort–Frankel method. The stability condition $\Gamma \leqslant \rho(D)$ is equivalent to the usual condition on Lax–Wendroff parabolic schemes that $D(\Delta t/(\Delta x)^2) \leqslant \frac{1}{2}$. The margnial stability found by Gottlieb and Gustafsson when $\Gamma = 1$ is a reflection of difficulties occurring when choosing the time step at the stability boundary. Chorin [1] has already pointed out the relationship of Dufort–Frankel to SOR.

This identification of the scheme proposed by Désidéri and Tannehill with MSOR and Dufort–Frankel for parabolic problems also suggests certain difficulties with the accelerated MacCormack scheme. First, time splitting schemes (see for example MacCormack and Paullay [7]) may no longer apply if $\bar{\omega} \neq 1$ since the scheme is not absolutely stable. Also the new scheme may require more storage. The regular MacCormack scheme can be programmed so that only one storage level is required without unnecessary evaluation of fluxes. The accelerated scheme, in common with all leap-frog-type methods, requires at least two levels of storage.

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